14/02/2

U.G., Sem IV MJC-05 Real Analysis (Cauchy's Root Test)

Let $\sum U_n$ be an infinite series of positive terms and let $\lim_{n\to\infty} U_n^{\frac{1}{n}} = l$, Then the series $\sum U_n$ (i) Converges if $l<1$;

(ii) diverges if $l>1$;

(iii) no conclusion if $l=1$.

Proof:- Since $U_n^{\frac{1}{n}} \to l$ as $n\to\infty$, therefore, from the definition of limit, it follows That there exists a positive integer m Such that $l-\epsilon < U_n^{\frac{1}{n}} < l+\epsilon$ ---- (1),

for all $n \geq m$ and $\epsilon > 0$.

Let $l<1$, Then we can choose ϵ such that $l+\epsilon<1$.

From the second inequality of (1), we have

$U_n^{\frac{1}{n}} < l+\epsilon$, for all $n\geq m$,

i.e. $U_n < (l+\epsilon)^n$, for all $n \geq m$.

Now $\sum(l+\epsilon)^n$ converges, as it is a geometric series with common ratio $l+\epsilon<1$; therefore, $\sum U_n$ also converges.

Again, let $l>1$, then we can choose ϵ such that $l-\epsilon>1$, from the first inequality of (1), we obtain $U_n^{1/n} > l-\epsilon$, for all $n\geq m$,

i.e. $U_n > (l-\epsilon)^n$, for all $n \geq m$.

But $\sum(l-\epsilon)^n$ diverges, as it is a geometric series with common ratio $l-\epsilon>1$; Therefore $\sum U_n$ also diverges.

This test fail when $\lim_{n\to\infty} U_n^{\frac{1}{n}} = 1$.